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MAKING THE SUPERCOMPACTNESS OF κ INDESTRUCTIBLE UNDER κ-DIRECTED CLOSED FORCING

BY

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ABSTRACT

A model is found in which there is a supercompact cardinal κ which remains supercompact in any κ-directed closed forcing extension.

We assume familiarity with the theory of supercompact cardinals (see Kanamori–Reinhardt–Solovay [2]) and with Silver's results (see Menas [4]) about preservation of supercompactness in certain upward Easton extensions.

A partial ordering P is *γ-directed closed* if whenever $D \subseteq P$ is directed and $\operatorname{Card} D < \gamma$ then there is a $p \in P$ with $d \leqq p$ for all $d \in D$. Clearly, it is consistent that κ-directed closed forcing can destroy the supercompactness of κ; if the G C H holds below κ, make $2^\kappa > \kappa^+$ in the standard way and κ won't be measurable, if $2^{\alpha^+} = \alpha^{+++}$ holds for every inaccessible $\alpha < \kappa$, make $2^{\kappa^+} = \kappa^{++}$ in the standard way and κ will still be measurable but no longer supercompact.

Menas proved that for κ supercompact, $\lambda > \kappa$, $\lambda^\kappa = \lambda$ there is a κcc partial order Q of power κ, such that upon forcing with Q followed by the standard partial ordering for making $2^\kappa = \lambda$, κ remains supercompact. We prove a strengthening of Menas' result.

THEOREM. *If κ is supercompact, then there is a κ*cc *partial ordering Q with* Card $Q = \kappa$, *such that in V^Q, κ is supercompact and remains supercompact upon forcing with any κ-directed closed partial ordering.*

(Since the partial ordering $\{\emptyset\}$ is κ-directed closed, the supercompactness of κ in V^Q is in fact derivable from the rest of the theorem.)

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Let $\mathrm{TC}(x)$ be the transitive closure of x. If $\kappa \leqq \lambda$ and U_λ is a supercompact ultrafilter on $[\lambda]^{<\kappa}$, let $M_{U_\lambda} = V^{[\lambda]^{<\kappa}}/U_\lambda$, and let $j_{U_\lambda} : V \to M_{U_\lambda}$ be the canonical embedding. (Will write $M_{U_\lambda} = M_\lambda$, $j_{U_\lambda} = j_\lambda$ without danger of confusion below.)

Menas [5] showed that if κ is supercompact then for every ordinal α there is an $f : \kappa \to \kappa$ and a supercompact ultrafilter U_λ on $[\lambda]^{<\kappa}$, for some λ, such that $(j_\lambda f)(\kappa) = \alpha$. The following lemma shows that some f works simultaneously in this way for all $x \in V$.

Lemma. *Let κ be supercompact. Then there is an $f : \kappa \to R_\kappa$ such that for every x and every $\lambda \geqq \mathrm{Card}\,\mathrm{TC}(x)$, there is a supercompact ultrafilter U_λ on $[\lambda]^{<\kappa}$ such that $(j_\lambda f)(\kappa) = x$.*

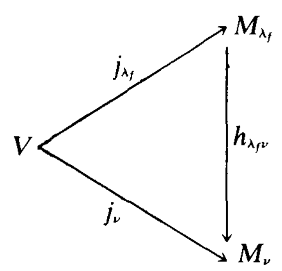
Proof. Otherwise there is, for each $f : \kappa \to R_\kappa$, a least ordinal λ_f such that there exists an x with $\mathrm{Card}\,\mathrm{TC}(x) \leqq \lambda_f$, with $\langle x, \lambda_f \rangle$ a counter-example to the lemma for f. Let ν be greater than all the λ_f's, and pick a supercompact ultrafilter U_ν on $[\nu]^{<\kappa}$.

Let $\Phi(g, \delta)$ be the statement that for some cardinal α, $g : \alpha \to R_\alpha$, and δ is the least ordinal for which there exists an x with $\mathrm{Card}(\mathrm{TC}(x)) \leqq \delta$ such that for no supercompact U_δ on $[\delta]^{<\alpha}$ does $(j_\delta g)(\alpha) = x$. Since $[M_\nu]^\nu \subseteq M_\nu$, we have that for each $f : \kappa \to R_\kappa$, $M_\nu \models \Phi(f, \lambda_f)$.

Let U_κ be the projection of U_ν on κ. There is thus an $A \in U_\kappa$ such that for each $\alpha \in A$ and each $f' : \alpha \to R_\alpha$ there is a $\lambda_{f'} < \kappa$ such that $\Phi(f', \lambda_{f'})$.

Define inductively an $f : \kappa \to R_\kappa$ ad follows. Suppose $\alpha < \kappa$ and $f_\alpha =_{\mathrm{def}} f \restriction \alpha$ has been defined. Then let $f(\alpha) = \varnothing$ unless $\alpha \in A$ and $f_\alpha : \alpha \to R_\alpha$. In this case there is an $x \in R_\kappa$ witnessing $\Phi(f_\alpha, \lambda_{f_\alpha})$; let $f(\alpha) = x_\alpha$ be such an x. We have the following relations: $(j_\nu \langle f_\alpha : \alpha \in A \rangle)(\kappa) = f$, $(j_\nu \langle \lambda_{f_\alpha} : \alpha \in A \rangle)(\kappa) = \lambda_f$, and $(j_\nu \langle x_\alpha : \alpha \in A \rangle)(\kappa) = (j_\nu f)(\kappa) =$ some x which witnesses $\Phi(f, \lambda_f)$ in M_ν, and hence in V.

Let U_{λ_f} be the projection of U_ν onto λ_f. We claim that $(j_{\lambda_f} f)(\kappa) = x$, which will contradict that x witnesses $\Phi(f, \lambda_f)$, proving the lemma. Namely, in the canonical commutative diagram



we have that $h_{\lambda_f\nu}$ is the identity on λ_f, $\operatorname{Card}\operatorname{TC}(x) \leqq \lambda_f$, whence $x \in M_{\lambda_f}$ and $h_{\lambda_f\nu}(x) = x$.

Thus $(j_{\lambda_f}f)(\kappa) = (h_{\lambda_f\nu})^{-1}((j_\nu f)(\kappa)) = (h_{\lambda_f\nu})^{-1}(x) = x$.

Proof of the Theorem. Let $f : \kappa \to R_\kappa$ be as in the lemma. The partial order Q of the theorem will be an upward Easton extension of length κ. For $\alpha \leqq \kappa$, denote the ordering corresponding to the first α stages in the iteration by Q_α; thus, $Q = Q_\kappa$. As we inductively define the Q_α's, ordinals λ_α, $\alpha < \kappa$, are also chosen. At a limit stage γ we take, as usual, Q_γ to be those sequences in the inverse limit of $\{Q_\beta : \beta < \gamma\rangle$ whose supports are Easton $\underset{\sim}{\text{sets}}$ of ordinals; let $\lambda_\gamma = \sup_{\beta<\gamma}\lambda_\beta$. To go from α to $\alpha + 1$, we put $Q_{\alpha+1} = Q_\alpha \otimes P_\alpha$, where P_α is defined as follows. P_α will be the term for the partial ordering $\{\varnothing\}$ unless

(1) for all $\beta < \alpha$, $\lambda_\beta < \alpha$, and

(2) $f(\alpha) = \langle P, \lambda \rangle$, where λ is an ordinal and P is a term in the forcing language of Q_α such that $\models_{Q_\alpha} P$ is an α-directed closed partial ordering.

When (1) and (2) hold we let $P_\alpha = P$, $\lambda_\alpha = \lambda$.

Let P be a term in the forcing language of Q_κ such that $\models_{Q_\kappa} P$ is a κ-directed closed partial ordering. We need to show that κ is supercompact in $V^{Q_\kappa \tilde{\otimes} P}$. Given a $\gamma \geqq \kappa$, we find a supercompact ultrafilter on $[\gamma]^{<\kappa}$ in $V^{Q_\kappa \tilde{\otimes} P}$. Let λ be a cardinal such that $\lambda > \operatorname{Card}\operatorname{TC}(P)$ and $\models_{Q_\kappa \tilde{\otimes} P} \lambda \geqq 2^{(\gamma^{<\kappa})}$. There is thus in V a supercompact ultrafilter U_λ on $[\lambda]^{<\kappa}$ such that $(j_\lambda f)(\kappa) = \langle P, \lambda \rangle$. Write $j_\lambda = j$.

In M_λ, $j\langle Q_\alpha : \alpha \leqq \kappa \rangle$ is a sequence of length $j\kappa$ which, by the method of its construction, must start with $\langle Q_\alpha : \alpha \leqq \kappa \rangle$; write then $j\langle Q_\alpha : \alpha \leqq \kappa \rangle = \langle Q_\alpha : \alpha \leqq j\kappa \rangle$, where $Q_{j\kappa} = j(Q_\kappa)$. Now κ satisfies $\underset{\sim}{\text{conditions}}$ (1) and (2) above (working $\underset{\sim}{\text{in}}$ M_λ with jf in place of f), so $Q_{\kappa+1} = Q_\kappa \otimes P$ and for $\kappa + 1 \leqq \delta < \lambda$, $Q_{\delta+1} = Q_\delta \otimes \{\varnothing\}$. The remainder of the iteration (from Q_λ to $Q_{j(\kappa)}$) is by construction a $\geqq \lambda$-closed forcing notion (in M_λ, and hence in V $\underset{\sim}{\text{since}}$ $[M_\lambda]^\lambda \subseteq M_\lambda$). Since $\lambda < j\kappa$, jP $\underset{\sim}{\text{is}}$ also a λ-closed forcing notion. Thus $Q_{j(\kappa)} \otimes jP$ is obtained by $\underset{\sim}{\text{following}}$ $Q_\kappa \otimes P$ by forcing with a λ-closed partial ordering which, living in $V^{Q_\kappa \tilde{\otimes} P}$, we will call R.

Silver's argument will now give us what we want. Namely, since jP is λ-directed closed over $Q_{j\kappa}$, and since $\lambda > \operatorname{Card}\operatorname{TC}(P)$, we may pick a condition $s \in jP$ with Silver's property that

$$(*) \qquad s \models_{Q_{j\kappa} \tilde{\otimes} jP} \forall p \in Q_\kappa \tilde{\otimes} P \qquad (p \in G \Leftrightarrow jp \in G).$$

Now work in $V^{Q_\kappa \tilde{\otimes} P}$. Recall that $\lambda \geqq 2^{(\gamma^{<\kappa})}$. Pick a sequence $r_0 \leqq r_1 \leqq \cdots \leqq r_\alpha \leqq \cdots (\alpha < \lambda)$ from R such that $r_0 = s$, and for any $X \subseteq [\gamma]^{<\kappa}$ there is an α such that r_α decides the statement $\{j\beta : \beta < \gamma\} \in jX$, furthermore, if $r_\alpha \Vdash \{j\beta : \beta <$

$\gamma\} \in jX$ and F is a choice function on X, then there are δ and ν such that $r_\delta \Vdash jF(\{j\beta : \beta < \gamma\}) = j\nu$. The statement of these requirements is unambiguous because of (*). We conclude, by the usual argument, that the set $\{X \subseteq [\gamma]^{<\kappa} : \exists \alpha\, r_\alpha \Vdash \{j\beta : \beta < \gamma\} \in jX\}$ is, in $V^{Q_\kappa \tilde{\otimes} P}$, a supercompact ultrafilter on $[\gamma]^{<\kappa}$, giving the theorem.

It is a corollary of the indestructibility of the supercompactness of κ under κ-directed closed orderings, that κ is also supercompact in any V^P such that P is a κ-directed closed Easton or upward Easton class partial ordering.

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